Germanium Resistance Thermometry in the Range 2.1 to 5.0 °K

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The resistances of three encapsulated, hermetically sealed, arsenic-doped germanium resistors from a commerical source have been measured at every 0.1 °K from 2.1 to 5.0 °K in highly stabilized liquid helium baths; values of temperature were derived from vapor pressure measurements associated with the liquid bath.

Ten similar germanium resistors of common origin have been calibrated from 2.1 to 5.0 °K in a calibration comparator apparatus that is similar to a calorimeter in design. Calibrations were performed against a resistor that had been previously referenced to temperature through a helium-4 vapor pressure calibration.

Next, three of the resistors that were calibrated in the comparator apparatus were recalibrated in a liquid helium bath (at 2.2, 3.0, and 4.2 °K) and results from the two methods agree within ± 1 millidegree. The resistance-temperature data from both methods have been fitted to the polynomial equation $\log_{10} R = \sum_{n=0}^{m} A_n \cdot (\log_{10} T)^n$ and the results of a computer program, which evaluates the coefficients, are presented.

1. Introduction

We have a group of impurity-doped germanium resistors that possess desirable thermometric properties at low temperatures—negative coefficient of temperature, good sensitivity and excellent reproducibility under thermal cycling [1].[2] This paper describes the calibration (in the range 2.1 to 5.0 °K) of the above-mentioned resistor group.

Several resistors have been calibrated directly against liquid helium-4 vapor pressures thus giving reference to temperatures from the "1958 He⁴ Scale of Temperatures" [2]. Since this method of calibration is costly in both time and refrigerant materials, a different procedure was utilized for extending the calibration over a greater number of resistors. An apparatus was constructed for comparison calibrations between "unknowns" and resistors that had been previously calibrated against helium-4 vapor pressures.

2. Calibration in a Liquid Helium Bath

2.1. Liquid Helium Bath

The apparatus, used for calibrating germanium resistors in a highly stabilized liquid helium bath, has been described in an earlier paper [1]. While the previous report was based upon calibrations in the vicinity of 4.2 °K only, no modification of the apparatus was necessary for expanding the calibration range to include 2.1 to 5.0 °K.

The bath consisted of about 8 liters of liquid helium which filled the spherical liquid helium storage Dewar (25 liters) to a height of 14 cm. This provided a liquid helium surface level that did not extend above the thermally insulating vacuum jacket of the vapor pressure bulb. The storage Dewar was then connected to a vapor pressure regulator [3] and the surface vapor pressure monitored by a mercury manometer. Small amounts of helium gas from a 1-liter filling can were bled into the jacketed vapor pressure thermometer. After the helium gas pressure in the thermometer equaled the controlled bath surface pressure, 1/3 of an atmosphere of helium gas was slowly metered into the bulb to condense approximately 0.5 cm³ of liquid. Since the thermometer bulb volume was 2 cm³, the fluid level was well below the top of the bulb and the remaining volume was sufficiently large to contain some additional condensate-condensation of gas from the vapor pressure thermometer tubing occurred when the vapor pressure of the helium bath was intentionally decreased during the course of a calibration.

The calibration from 2.1 to 5.0 °K was procedurally divided into three stages. Obvious checks were made to ascertain that there was appropriate electrical continuity of resistor leads and no undesired grounding in the resistance measuring circuit before the resistors were lowered into helium bath. Then the Dewar bath was cooled to appropriate temperatures

by pumping through the vapor pressure regulating manostat and calibrations were performed at every 0.1 °K in the range 4.2 to 2.6 °K. Next, the Dewar bath was warmed to the helium boiling point (by bleeding helium gas into the Dewar) and a second liquid helium transfer made. The second transfer was necessary since a height of only 3.2 cm of liquid helium remained in the Dewar after the calibration at 2.6 °K was finished. The bath was then cooled directly to 2.5 °K and subsequent calibrations performed at the five lower temperature points — 2.5, 2.4, 2.3, 2.2, and 2.1 °K. Thirdly, the Dewar bath was warmed to 4.2 °K, a necessary third liquid helium transfer made and the calibration performed in the range 4.2 to 5.0 °K. (Vapor pressures that exceed atmospheric pressure were obtained by closing off the manostat and allowing the Dewar vapor pressure to build up to a desired value at which time it was again manostatically controlled. The normal evaporation rate within the Dewar provided the gas for the increased pressure.)

2.2. Vapor Pressure Thermometry

The vapor pressure measuring techniques, used in these measurements, were consonant with procedures that we have previously reported [4]. Since a temperature gradient exists at the surface of the "constant temperature" liquid bath, the vapor pressure thermometer tubing must be thermally insulated from the colder surface to avoid spurious effects caused by "cold spots" [4,5]. Consequently, the thermometer tubing was surrounded by a vacuum jacket — the tubing length that otherwise would be in contact with the liquid surface — thus insuring that the vapor pressure thermometer indicated a vapor pressure that was associated with the bulk liquid. Vapor pressure measurements of both the helium surface and the thermometer bulb were routinely performed and recorded. In a previous paper [4], we have listed the observed differences in these two measurements for tenth-degree intervals from 2.1 to 4.2 °K. Table 1 lists typical comparisons derived from the calibration data that are the basis for this paper. The differences between the two vapor pressures, P_J and P_S, are not unique since the pressure and temperature gradients, at the liquid helium surface, appear to be dependent upon at least two conditions — the geometry of the contained liquid helium and the Dewar evaporation rate. Nevertheless the qualitative change in magnitude of the surface temperature gradient (ΔT) as a function of the helium bath temperature is supported by the results of many experimental "runs" in our laboratory. Additionally, as we have previously reported [4], calibrations of germanium resistors across the helium—λ point have been "smooth" when temperatures in the helium-I region were derived from pressures measured in conjunction with the jacketed vapor pressure thermometer. We have been unable to obtain a "smooth" calibration across the λ-point when temperatures are derived from surface vapor pressure measurements; as might be suspected, tem-

peratures derived from either method are satisfactory in the temperature region of helium-II provided the effects of helium film creep do not enter as a variable.

In the range 4.2 to 5.0 °K, the vapor pressure bulb required periodic additions of helium gas since, as the vapor pressure was increased, some of the previously condensed liquid helium must be evaporated to supply the required helium gas in the vapor pressure thermometer tubing. The inverse operation was necessary as temperatures were reduced below 4.2 °K i.e., liquid helium was occasionally removed from the bulb to avoid an overfilling of the bulb.

TABLE 1. *Vapor pressure measurements in a "constant" temperature liquid helium-4 bath*

Temperature	$P_J{}^a - P_S{}^b$	$\Delta T{}^c$	Temperature	$P_J{}^a - P_S{}^b$	$\Delta T{}^c$
°K	mm Hg	mdeg	°K	mm Hg	mdeg
5.0	0.43	0.4	3.5	0.84	2.0
4.9	.38	.4	3.4	.76	1.9
4.8	.66	.7	3.3	.71	2.0
4.7	.71	.7	3.2	.66	2.0
4.6	.86	1.0	3.1	.61	2.0
4.5	.97	1.1	3.0	.58	2.1
4.4	.99	1.2	2.9	.58	2.4
4.3	1.02	1.4	2.8	.56	2.6
4.2	0.91	1.3	2.7	.48	2.5
4.1	1.07	1.6	2.6	.48	2.8
4.0	0.97	1.6	2.5	.43	2.8
3.9	.91	1.6	2.4	.38	2.9
3.8	.90	1.7	2.3	.38	3.3
3.7	.90	1.8	2.2	.33	3.4
3.6	.81	1.8	2.1	.00	0.0

[a] P_J is the helium vapor pressure of the jacketed thermometer.
[b] P_S is the helium vapor pressure of the bath surface.
[c] ΔT (the temperature equivalent of the pressure difference, $P_J - P_S$) is equal to $\frac{(P_J - P_S)}{\frac{dP}{dT}}$. $\frac{dP}{dT}$ is obtained from the "1958 He⁴ Scale of Temperatures" [2].

In the range 4.8 to 5.0 °K there were a few instances when the helium gas oscillated in the jacketed vapor pressure tube. These oscillations [6] were indicated by: (1) Thermometer vapor pressures greatly exceeded surface vapor pressures of the bath after surface pressure control had been established for several hours; (2) the evaporation rate of the helium bath, as measured with a flow meter, was greatly increased (by as much as a factor of three) but returned to its normal value when the bulb was evacuated; and (3) large oscillations were observed in the mercury manometer. Reliable measurements were obtained in these instances by altering the liquid bath temperature to a value where oscillations were not encountered.

Resistance determinations, at each stabilized temperature, were measured potentiometrically; the electrical power applied to a resistor generally did not exceed 0.02 μW, but at temperatures above 4.6 °K, it was possible to apply 0.1 μW without causing appreciable joule heating in the resistors.

2.3. Discussion

The calibration at each temperature in the range 2.1 to 4.2 °K required approximately 3 hr — ½ to 1 hr for cooling the bath to the desired surface vapor pressure, 2 hrs for the bath to stabilize and ½ hr for performing electrical measurements. In one range above the

boiling point, 4.7 to 5.0 °K, it required about 24 hrs to increase the bath temperature by 0.1 °K. Therefore, it was not unusual to expend three weeks in performing a calibration at every 0.1 °K from 2.1 to 5.0 °K. The fact that at least three liquid helium transfers were necessary to cover the range made this calibration expensive as well as time consuming. For these reasons, an isothermal comparator was constructed for calibrating a large number of resistors more efficiently.

3. Calibration in an Isothermal Comparator

3.1. Comparator

The comparator design and construction is shown in figure 1. The copper top A of the outside can B is 6.04 cm in diameter and has an internal extended sleeve A' (1.90 cm O.D., 2.22 cm long, and 1.27 cm wall thickness). The outside can B is mainly brass and has the dimensions of 6.67 cm diam, 21 cm long, and 0.159 cm wall thickness; the upper 1.27 cm of the can is a piece that was machined to receive the cylindrical tongue of top A. Wood's metal is used to solder the copper top to the brass can. The bottom of the brass can C is a brass disk that has been silver soldered to the outside cylinder. During a calibration, the can B is contained in a liquid bath whose liquid level is well above the top A. The thermal anchor post T, 0.477 cm copper rod, is screwed into A. Three Teflon rods, each of 0.32 cm diam and 3.2 cm long, are equally spaced on the end of cylinder A' and screwed into the wall thickness. A 0.64 cm perforated stainless steel tube E was silver soldered to the copper top A; its opposite end was soft soldered to the top of the 2.5 cm solid copper cylinder J. Twelve 0.36 cm diam holes N were drilled at an angle of 45 degrees to the axis of J and to a depth of 1.12 cm, which is sufficiently deep to contain the length of a germanium resistor encapsulation. The inner copper can K, 6.4 cm outside diameter, 9.5 cm long, screws onto the threaded surface of the shoulder M. K serves as a radiation shield for the portion of the cylinder that contains the resistors. Several 0.16 cm holes O were drilled through K to enhance evacuation of the volume within K. F and F' (0.64 cm, thin wall, stainless steel tubings), were soft soldered through A and serve respectively, as a helium gas filling or evacuation line and conduit for lead wires to the resistors and heater coil. A total of 14 leads (#38 A. W. G. formvar coated, double nylon insulated, copper wire) are used as connecting wires—4 pairs as potential leads to the germanium resistors that are involved in a calibration, 1 pair for current leads to the series connected resistors and 1 pair each for the sensor and heater coil. For the heater coil, #36 A.W.G. manganin wire was noninductively wound on the bobbin H to provide a total resistance of 38,000 Ω at room temperature.

The seven pairs of connecting wires were conducted down from the Dewar cap through F' (see fig. 2) and wound around the thermal anchor T—prior to wrapping
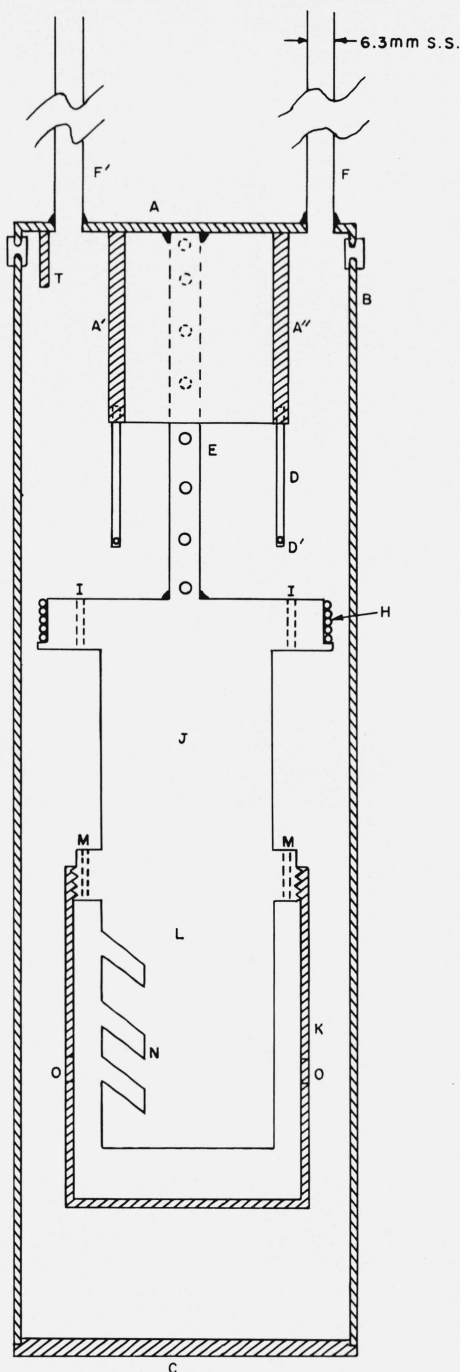
FIGURE 1. *Design of comparator—the low temperature component of apparatus employed in calibrating germanium thermometers from 2.1° to 5.0 °K.*

around T, a small loop was made to allow for thermal contraction of the leads within the tube F'. Next, about 60 cm of the leads were spirally wound about the thermal anchor A' and afterwards, coatings of G.E. varnish #7031 were applied to the wires and surfaces of T and A' to enhance the thermal contact between
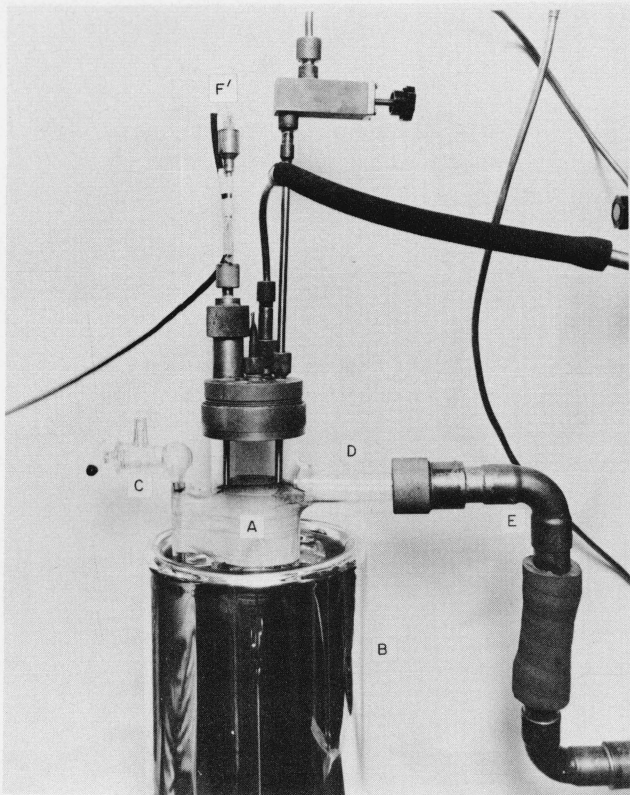
FIGURE 2. *A view of the upper portion of the cryostat that contains the calibration comparator.*

the lead wires and thermal anchors. This treatment will greatly reduce the heat leaks along the wires from room temperatures to the resistors. Additional lengths of wire were then spiralled around the three Teflon posts D and strung through holes D' drilled near the end of the Teflon rods. One pair of leads was separated from the group and connected to the heater coil. The remaining 6 pairs were conducted through I, 0.159 cm holes that had been previously lined with Teflon, and thence spiralled around J where #7031 varnish was also applied. Next, the wires were then led through M, 0.159 cm holes also lined with Teflon, wrapped once around L and thence soldered to the germanium resistor leads with Wood's metal. The soldered connections are enclosed by spun glass spaghetti to avoid electrical shorting or grounding of the leads.

In use, the vacuum can B and its top A are exposed to a bath temperature that is less than 2 °K, while the cylindrical block J is maintained at calibration temperatures that exceed 2 °K. Thus, without proper precautions, large undesired heat-flows could occur and cause: (i) a temperature instability of the block J with resultant gradients; (ii) abnormally large evaporation rates of the surrounding liquid helium bath; and (iii) a lack of thermal equilibrium between the resistors that are to be calibrated. The comparator has been designed to reduce and control the flow of heat toward

and away from J. (1). Radiant transfer from J or K to the outer can B is small. (2). The continuous evacuation of B minimizes heat transfer through gas conduction and convection. (3) Heat conduction through the leads and mechanically supporting members has been reduced.

There are several other component systems that are used in conjunction with the comparator. Temperature control of L or J (fig. 1) is effected by an automatic electronic system. A germanium sensor, mounted in a cavity N of L (fig. 1) serves as one leg of a Wheatstone bridge. Any unbalance of the bridge is amplified and fed into a controller which in turn regulates current flow through the heater H (fig. 1).

Shown in figure 2, is the upper portion of the cryostat that consists of a glass Dewar A (7.0 cm I.D. and 100 cm long), which contains the liquid helium in which the comparator is immersed, and an external Dewar that contains liquid nitrogen for thermally shielding the liquid helium Dewar. The figure shows a vacuum tight Dewar cap that accommodates the two tubes, which lead to the comparator (tubes F and F' of fig. 1), and permits filling of the Dewar with liquid helium. Also shown, is the Dewar's horizontal sidearm D that connects to a vacuum pumping line E—by "pumping" on the vapor pressure through this line, helium bath temperatures of less than 2 °K are achieved. The vertical tubing, which includes the vacuum valve, leads from an external "high vacuum" pumping system to the insulating vacuum space of the comparator; electrical lead ingress to the comparator is afforded by the tubing on the left. All of the apparatus, with the exception of the helium vapor pressure pumping system, is located within a shielded room that isolates the equipment from local rf interference [7].

3.2. Comparator Operating Procedure

The resistors that were involved in a calibration were mounted in the copper block J (fig. 1) in a medium of Apiezon stopcock grease to enhance thermal contact. The outer can B was soldered in place with Wood's metal and the 14 electrical leads then checked for electrical continuity. After appropriate leak detection the comparator was mounted in the cryostat, a small amount of helium gas was bled into the vacuum space (within can B) and the apparatus precooled to liquid nitrogen temperatures. Next, liquid helium was transferred into the inner Dewar, resistor lead continuity checks performed again, and then the liquid helium bath cooled to 1.5 °K. After the helium exchange gas was sufficiently evacuated, the resistance of the Wheatstone bridge was set to correspond to the 2.1 °K resistance value (determined in an initial calibration) of the temperature controlling sensor. The three control dials—proportional band, rate time and reset—of the servocontrol unit were set to obtain adequate temperature control of the comparator. Proper settings of the three controls result in a temperature control as fine as 1×10^{-4} °K at 2.1 °K. This has been evidenced by repetitive resistance deter-

minations for resistor #1 over a period of 30 min. Generally, once a good setting has been obtained at the lowest calibration temperature, the dials need not be changed for the remainder of the calibration.

In establishing any given calibration control point, a value of resistance was dialed on the Wheatstone bridge and sufficient time was allowed for the recorder pen of the temperature controlling unit to reach the control or zero position. (Generally about 10 min was required to attain temperature stability after an increase of 0.1 °K in the temperature of the comparator.) Initial measurements on resistor #1 often indicated a deviation of approximately several millidegrees from a desired calibration temperature. In this event appropriate corrections were made on the Wheatstone bridge settings, five additional minutes allowed for equilibrium and measurements were again made across #1. This process was repeated until the resistance of #1 indicated a value that was within ± 0.5 mdeg of the desired calibration temperature.

4. Germanium Resistance Thermometers

For all of the germanium resistors, whose calibrations are being reported in this paper, we have a thermal cycling history. That is, the resistors have been thermally cycled from room temperature to 4.2 °K and systematic calibrations of resistance and temperature performed at 4.2 °K. The methods of cycling and calibration have been previously described [1], and the reproducibilities of particular resistors were stated. The previous resistor identification symbols will apply to precisely the same resistors in this report. All of the resistors are four lead models and are hermetically encapsulated in platinum containers filled with helium gas. However, the calibrator design and method of mounting resistors in the comparator are such that a leaking encapsulation will probably not affect the general validity of a calibration. Most of the resistors are similar in that their values of resistance are comparable at 4.2 °K — this probably means that their "dopings" are similar and consequently, that similar equations will represent the resistance temperature calibration data.

5. Experimental Results

5.1. From the Stabilized Helium Bath

Resistors 1, 2, and 3 (reproducible to approximately 1 mdeg under thermal cycling [1]) have been calibrated simultaneously in the "constant temperature" helium bath from 2.1 to 5.0 °K at intervals of 0.1 °K; the results are listed in table 2. A log-log plot of resistance versus temperature for #1, #2, and #3, shown in figure 3, suggests the possibility of representing the calibration data by a polynomial of the form

$$\log_{10} R = \sum_{n=0}^{m} A_n \, (\log_{10} T)^n.$$

TABLE 2. *Calibrated resistances obtained in the highly stabilized liquid helium baths*

Resistor #1	#2	#3	Temperature [a]
Ω	Ω	Ω	°K
13281.	12435.	13739.	2.0993
11749.	10982.	12064.	2.2013
10528.	9821.0	10740.	2.2995
9491.6	8839.0	9611.0	2.3984
8588.0	7979.0	8632.0	2.5003
7841.0	7271.7	7832.3	2.5994
7106.4	6572.9	7042.0	2.7130
6611.8	6108.0	6519.0	2.8005
6119.1	5639.4	6000.0	2.8996
5681.2	5227.1	5537.5	2.9986
5287.2	4855.8	5128.3	3.0997
4938.8	4527.8	4766.3	3.1997
4625.8	4232.8	4443.0	3.2997
4345.1	3968.7	4155.1	3.3996
4090.2	3729.9	3894.6	3.4999
3859.6	3513.4	3660.6	3.5996
3650.4	3317.0	3448.2	3.6996
3457.4	3136.9	3253.8	3.7999
3279.7	2970.6	3075.3	3.9007
3118.8	2820.7	2914.7	4.0000
2972.0	2683.4	2768.6	4.0981
2829.8	2551.8	2627.8	4.2000
2699.2	2430.2	2498.7	4.3014
2579.6	2318.9	2381.1	4.4011
2468.6	2216.2	2272.1	4.5002
2363.2	2119.0	2169.7	4.6009
2266.3	2029.0	2075.1	4.6998
2173.4	1942.8	1985.4	4.7999
2086.6	1862.7	1901.4	4.9004
2005.4	1787.9	1823.1	5.0005

[a] Temperatures are derived from the "1958 He⁴ Scale of Temperatures" [2] using values of vapor pressure that were obtained with a jacketed vapor pressure thermometer.
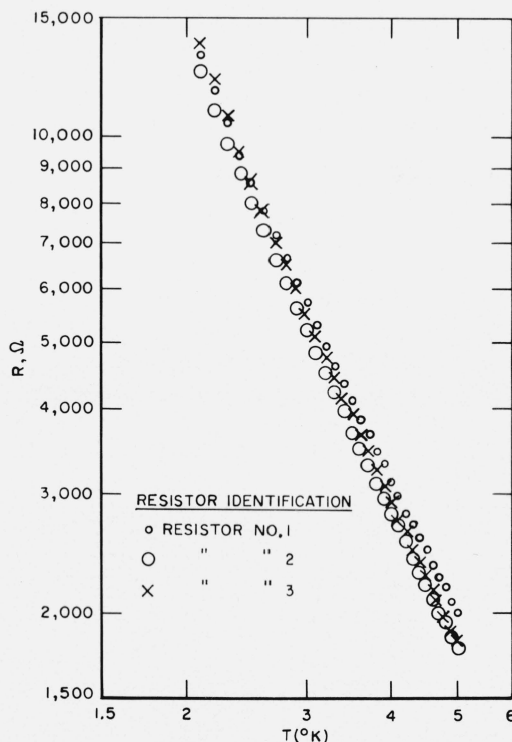

FIGURE 3. *A plot of the resistance-temperature calibration data for resistors 1, 2 and 3.*

Temperatures were derived from liquid helium-4 vapor pressures.

Consequently, the resistance temperature calibration data (obtained from the bath calibration) were programmed on a computer for a least squares fitting to the above polynomial. The program was "cascaded" so that a succession of least squares fitting was performed for upper limits of m that extended consecutively from 3, . . ., 8. The results require discussion since there are at least two ways of considering the representation of the calibration data.

(a) Except for the effect of spurious errors, the limiting accuracy of a calibration could possibly approach ± 0.001 °K. This would probably occur at the lowest temperatures and would be mainly associated with the accuracy of our helium vapor pressure measurements.

(b) If, however, one feels that the 0.001 °K assessment of error is high or might be smoothed out in the polynomial fitting, it is not obvious as to how far one should proceed with the polynomial fitting.

Because of these two considerations we felt compelled to present, in this paper, results which would be in accordance with either (a) or (b) of the preceding paragraph. In the computer analyses for the three resistors, the polynomial $\log_{10} R = \sum_{n=0}^{3} A_n (\log_{10} T)^n$ could possibly represent the data within the limits of condition (a). However there exists an obvious systematic deviation between the fitted function (R_{calc}, T) and the input calibration data (R_{data}, T_{data}). This is apparent in figure 4 where points of inflection occur near 2.8, 3.6, and 4.4 °K. While the general conclusion from figure 4 might be the indication that the input data has been adequately fitted within the limits of experimental errors, another possibility is suggested—the data is more consistent than ± 0.001 °K and the function that would represent the data is not adequately approximated by $\log_{10} R = \sum_{n=0}^{3} A_n (\log_{10} T)^n$. Consequently we have continued the analysis to higher order polynomials. Table 3 contains the evaluated polynomial coefficients for resistor #1; a double precision computer program prepared by
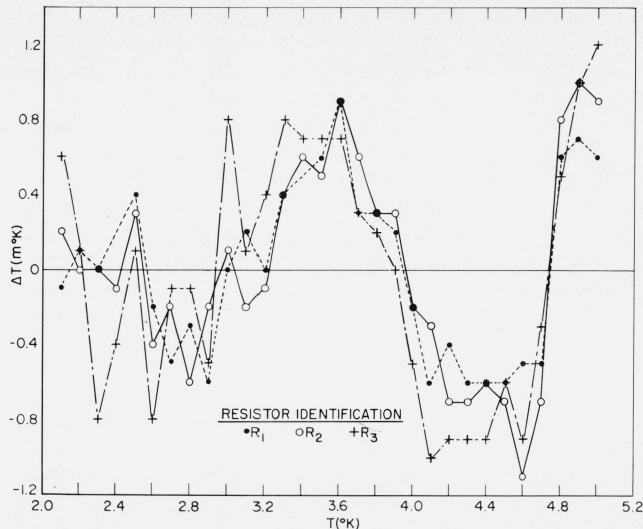

FIGURE 4. *Differences between the fitted functions,* $\log R = \sum_{n=0}^{3} A_n (\log T)^n$ *and the calibration data for resistors 1, 2 and 3.*

(a) The residuals $\Delta T = (R_{Gen.} - R_{Data})dT/dR$ is the temperature equivalent of $(R_{Gen} - R_{Data})$ which is the difference between the fitted polynomial function evaluated at T_{Data} (R_{Gen}) and the resistance value of the calibration (R_{Data}, T_{Data}). dR/dT is the temperature coefficient of resistance.

(b) The residual plots apply to the three resistors [#1 (.), #2 (0) and #3 (+)]; the calibration data for each resistor have been individually fitted to third order polynomials.

the National Bureau of Standards Statistical Engineering Laboratory was used to overcome the roundoff error characteristic of fitting high order polynomials, and in this case the roundoff error was several orders of magnitude smaller than other experimental uncertainties.

As higher order polynomials are used, the systematic differences—between the fitted polynomial and the input data—gradually diminish up to the 7th order where apparently random residuals result. Figure 5 is the plot of these residuals resulting from fitting the seventh order polynomial for resistors 1, 2, and 3. For $n = 7$ of table 3, it is apparent that the

TABLE 3. *Coefficients of the polynomial,* $\log_{10} R = \sum_{n=0}^{m} A_n (\log_{10} T)^n$, *fitted to the calibration data of Resistor #1*

Polynomial order	$m=3$		$m=4$		$m=5$		$m=6$		$m=7$		$m=8$	
Coefficient	Coef.	δC [a]	Coef.	δC [a]	Coef.	δC [a]	Coef.	δC [a]	Coef.	δC [a]	Coef.	δC [a]
A_0	5.202	(0.0023)	5.180	(0.0112)	5.380	(0.0396)	5.685	(0.198)	2.219	(0.735)	[1.86]	(2.63)
A_1	−4.249	(.0145)	−4.062	(.0922)	−6.160	(.411)	−9.995	(2.48)	40.873	(10.72)	[46.9]	(43.7)
A_2	3.304	(.0287)	2.732	(.279)	11.348	(1.68)	31.136	(12.74)	−284.78	(66.3)	[−329.]	(316)
A_3	−1.593	(.0185)	−.8355	(.369)	−18.206	(3.37)	−71.870	(34.41)	1004.6	(225)	[1186]	(1291)
A_4			−.3687	(.179)	16.846	(3.33)	[b] [97.546]	(51.61)	−2076.8	(454)	[−2538]	(3268)
A_5					−6.715	(1.30)	[−70.554]	(40.77)	2533.5	(542)	[3278]	(5248)
A_6							[20.767]	(13.26)	−1692.1	(356)	[−2435]	(5221)
A_7									477.6	(99.3)	[897]	(2943)
A_8											[−103]	(720)
Std. Dev.	1.12×10^{-4}		1.06×10^{-4}		7.4×10^{-5}		7.2×10^{-5}		5.1×10^{-5}		5.2×10^{-5}	

[a] δC is the standard error of a coefficient.
[b] The brackets indicate that the enclosed coefficient is poorly defined; that is, the ratio of the coefficient to its standard error is less than 2.
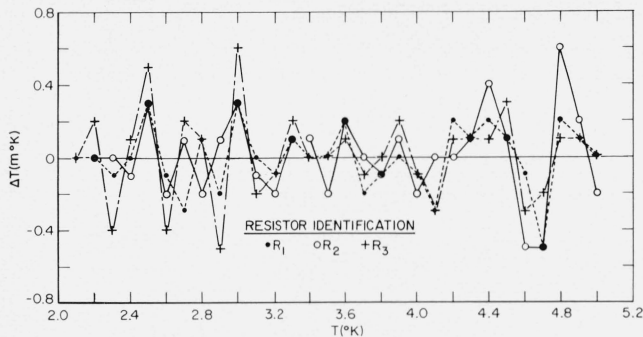
FIGURE 5. *Differences between the fitted functions, $\log R = \sum_{n=0}^{7} A_n$ $(\log T)^n$ and the calibration data for resistors 1, 2 and 3.*

(a) See (a) of figure 4.
(b) The residual plots apply to the three resistors [#1 (.), #2 (0) and #3 (+)]; the calibration data for each resistor have been individually fitted to seventh order polynomials.

first coefficient is the least defined coefficient of the column. Since random residuals, exhibited in figure 5, were achieved by the seventh degree polynomial and the coefficients of the eighth degree are poorly determined, we did not attempt to extend the fitting to higher order polynomials. (It does seem, for resistor 1 in table 3, that odd order polynomials provide a better functional fit than even orders. This observation is generally applicable to other germanium resistor calibrations between 2 and 5 °K and will be discussed later in the paper.)

In comparing figures 4 and 5, one observes that not only has the systematic deviation of figure 4 been removed by progressing to the polynomial fitting portrayed in figure 5, but also, the values of standard deviation that are listed in the last row of table 3 have been reduced by a factor of two. Despite this, one should be cautious about inferring: either (a), the calibration data were valid to ± 0.0005 °K; or (b), the polynomial fitting is completely satisfactory; or (c), conclusions concerning the He⁴ vapor pressure scale against which calibrations were performed. These items will require further clarification.

Since resistor #1 was to serve as a basis for calibrating additional resistors, a decision had to be made regarding its calibration representation. Although it is somewhat arbitrary, we decided to use the 7th order polynomial because it might have the effect of smoothing out experimental errors in resistor #1's calibration. The seventh order coefficients (table 3) were used to generate R, $\dfrac{dR}{dT}$ and $\dfrac{d^2R}{dT^2}$ for resistor #1 at intervals of 0.0001 °K from 2.1 to 5.0 °K; this generated table is the basis for calibrating additional resistors against resistor #1. It is pertinent to state that both the generated $\dfrac{dR}{dT}$ and $\dfrac{d^2R}{dT^2}$ did not exhibit a sign change in the range of generation; that is, the fitted seventh order polynomial is a smooth function in the range of use.

In general the polynomial fittings of resistors #2 and #3 produce results that parallel the findings for resistor #1. This is indicated by the plots of residuals for resistors #2 and #3 in figures 4 and 5.

5.2. Comparison Calibrations With Reference to Resistor #1

Ten resistors have been calibrated in the isothermal comparator with reference to the calibrated resistor #1 and the data are listed in tables 4 to 8. In each of the indicated runs, three "unknowns" were calibrated against resistor #1 (the data for the third resistor is not included in table 4). As a control on the stability of the calibrations, resistor I was measured in each of the five calibration runs. Thus there appears in each table the measured resistances of resistors #1 and I and two unknowns (one unknown in table 4). Temperatures, which are assigned to the resistance values of resistor #1, are taken from the Resistance-Temperature generation that was derived from a seventh order polynomial fitting (sec. 5.1).

It is also readily noted that particular calibration temperatures, in the 5 separate runs, are generally realized within a few millidegrees. This was intentional and eliminates interpolation inaccuracies that

TABLE 4

Run 3

Calibration data for Resistor I and Resistor L

R_1	$T^{\,a}$	R_1	R_L
Ω	°K	Ω	Ω
13241	2.1017	17117	15178
11754	2.2010	14905	13304
10584	2.2945	13192	11849
9515.4	2.3960	11652	10530
8604.7	2.4985	10365	9420.9
7842.2	2.5991	9303.5	8499.4
7107.7	2.7125	8295.7	7620.1
6610.6	2.8008	7624.3	7030.5
6119.9	2.8992	6969.6	6453.0
5681.7	2.9987	6392.7	5941.1
5285.5	3.1001	5877.7	5483.3
4940.6	3.1990	5433.0	5085.5
4625.3	3.2999	5033.2	4726.3
4343.9	3.4000	4680.2	4407.7
4090.7	3.4996	4365.1	4121.8
3859.0	3.6000	4081.3	3863.6
3457.2	3.7999	3595.4	3419.0
3279.3	3.9009	3383.8	3224.3
3118.6	4.0000	3194.4	3049.0
2972.2	4.0976	3023.1	2890.0
2830.4	4.1997	2859.0	2737.3
2698.7	4.3019	2707.5	2596.1
2579.8	4.4010	2572.0	2469.2
2467.7	4.5010	2445.6	2350.3
2362.5	4.6013	2327.3	2239.0
2266.2	4.6993	2220.2	2137.7
2173.3	4.8001	2117.5	2040.3
2086.7	4.9002	2021.9	1949.9
2005.5	5.0002	1933.6	1865.6

^{a} T is the temperature associated with the adjacent value of resistance for R_1, which was previously calibrated against T_{58} [2].

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TABLE 5

Run 5

Calibration data for Resistor H, Resistor M, and Resistor I

R_1	T [a]	R_H	R_M	R_I
Ω	$^\circ K$	Ω	Ω	Ω
13246.0	2.1014	14596	13819	17121
11742.0	2.2018	12801	12165	14882
10579.0	2.2949	11425	10894	13182
9507.0	2.3968	10173	9725.7	11639
8609.8	2.4979	9130.1	8754.7	10371
7840.1	2.5994	8245.1	7924.5	9298.5
7107.0	2.7127	7402.4	7137.8	8292.8
6609.6	2.8010	6845.5	6607.1	7620.4
6121.2	2.8990	6293.9	6088.5	6969.8
5682.7	2.9985	5804.0	5623.7	6392.8
5287.8	3.0995	5366.1	5208.4	5879.1
4941.5	3.1988	4981.9	4843.3	5433.9
4626.7	3.2995	4636.0	4513.9	5034.3
4344.7	3.3997	4329.6	4219.9	4681.1
4091.9	3.4991	4056.7	3956.7	4367.1
3859.6	3.5998	3806.8	3716.1	4082.2
3650.4	3.6993	3584.0	3500.1	3828.0
3458.5	3.7992	3379.7	3302.4	3597.3
3280.0	3.9005	3190.9	3118.9	3384.7
3118.9	3.9998	3020.9	2954.1	3194.5
2973.4	4.0968	2869.1	2805.7	3024.8
2830.1	4.1999	2718.8	2659.8	2858.5
2699.1	4.3015	2582.8	2526.6	2707.9
2579.1	4.4015	2459.8	2406.0	2571.5
2468.3	4.5004	2345.7	2294.1	2446.7
2363.9	4.5999	2238.3	2188.3	2328.9
2266.7	4.6988	2139.6	2091.0	2221.1
2173.6	4.7997	2044.6	1897.4	2117.8
2086.4	4.9006	1956.5	1910.2	2022.0
2005.1	5.0007	1874.4	1829.3	1933.3

[a] T is the temperature associated with the adjacent value of resistance for R_1, which was previously calibrated against T_{58} [2].

TABLE 6

Run 6

Calibration data for Resistor C, Resistor G, and Resistor I

R_1	T [a]	R_C	R_G	R_I
Ω	$^\circ K$	Ω	Ω	Ω
13244	2.1015	14995	14431	17120
11747	2.2015	13165	12649.5	14889
10580.5	2.2949	11754	11276.5	13185.5
9513.5	2.3962	10471	10031	11649
8611.0	2.4977	9395.1	8986.7	10372.3
7841.3	2.5993	8486.1	8105.1	9300.4
7105.7	2.7129	7623.0	7268.8	8291.2
6612.7	2.8004	7048.8	6713.2	7626.1
6121.6	2.8989	6481.7	6165.1	6971.2
5681.5	2.9988	5976.7	5676.4	6390.3
5287.1	3.0997	5527.7	5243.6	5879.7
4939.7	3.1993	5132.5	4862.5	5432.2
4625.7	3.2998	4777.7	4520.8	5033.1
4344.8	3.3997	4463.5	4217.5	4680.6
4090.9	3.4995	4180.6	3945.2	4366.8
3859.7	3.5997	3924.0	3699.0	4082.5
3650.0	3.6995	3692.5	3476.8	3827.9
3457.9	3.7995	3482.0	3275.9	3597.2
3280.6	3.9001	3287.6	3089.0	3385.6
3119.1	3.9997	3112.2	2921.2	3194.9
2972.1	4.0977	2953.4	2769.9	3023.3
2829.8	4.2001	2799.9	2623.6	2858.0

[a] T is the temperature associated with the adjacent value of resistance for R_1, which was previously calibrated against T_{58} [2].

TABLE 7

Run 7

Calibration data for Resistor F, Resistor J, and Resistor I

R_1	T [a]	R_F	R_J	R_I
Ω	$^\circ K$	Ω	Ω	Ω
13238	2.1019	15290	15228	17092.5
11745	2.2016	13364	13293.5	14873.5
10579.5	2.2949	11880.5	11807.0	13173.5
9509.8	2.3965	10532.0	10455.4	11632.6
8608.0	2.4981	9409.7	9336.2	10355.8
7843.9	2.5989	8472.0	8398.6	9297.2
7108.4	2.7124	7578.0	7509.6	8286.5
6609.3	2.8010	6979.5	6912.6	7615.6
6119.5	2.8993	6395.9	6332.4	6960.7
5680.5	2.9990	5880.2	5820.4	6384.4
5287.2	3.0996	5422.9	5366.3	5875.4
4938.4	3.1997	5018.7	4965.9	5427.2
4624.8	3.3001	4661.1	4611.4	5029.8
4345.4	3.3994	4342.6	4296.0	4680.1
4089.7	3.5000	4055.4	4011.6	4362.6
3859.9	3.5996	3799.7	3760.0	4081.6
3649.8	3.6996	3567.2	3530.2	3825.6
3457.3	3.7998	3355.6	3322.3	3594.6
3279.7	3.9006	3162.5	3131.4	3382.9
3118.7	3.9999	2987.0	2959.6	3192.4
2972.1	4.0977	2830.8	2805.2	3022.7
2830.1	4.1999	2678.2	2655.6	2856.5
2699.4	4.3013	2539.8	2519.1	2707.0
2579.6	4.4011	2414.4	2396.0	2571.2
2467.9	4.5008	2296.9	2280.6	2445.1
2363.1	4.6007	2187.8	2173.4	2327.9
2266.4	4.6991	2088.0	2075.3	2219.9
2173.0	4.8004	1992.0	1981.4	2116.6
2086.8	4.9001	1903.5	1894.2	2022.1
2005.1	5.0007	1820.7	1813.2	1932.9

[a] T is the temperature associated with the adjacent value of resistance for R_1, which was previously calibrated against T_{58} [2].

might otherwise arise in comparing the five individual calibrations of resistor I. Table 6 possesses no calibration data above 4.2 °K because obvious experimental difficulties,[3] which rendered the data invalid, occurred at calibration temperatures above 4.2 °K.

The calibration data for resistor I will be treated separately in a later section of the paper. The data for the other nine resistors has been subjected to computer fitting and analysis for a polynomial of the form

$$\log_{10} R = \sum_{n=0}^{m} A_n (\log_{10} T)^n.$$

The determined coefficients and their uncertainties are given in table 9. Again it is to be noted that: the calibrations of resistors in run VI cover a shorter temperature interval (2.1 to 4.2 °K); and the polynomial fitting of this data would be expected to produce coefficients that are significantly different from the coefficients representing the calibration of other resistors.

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[3] The experimental difficulty applies to run 6 (table 6) only, and was connected with the determination of the current that flowed through the resistor circuit during the 4.3 to 5.0 °K calibration.

TABLE 8

Run 9

Calibration data for Resistor K, Resistor E, and Resistor I

R_1	T [a]	R_K	R_E	R_I
Ω	°K	Ω	Ω	Ω
13251	2.1011	14099	16802	17134
11750	2.2013	12377	14658	14889
10603	2.2928	11073	13043	13215
9519.2	2.3956	9845.9	11541	11645
8605.8	2.4984	8827.4	10291	10366
7845.3	2.5987	7979.3	9257.5	9302.0
7108.5	2.7124	7167.2	8273.2	8293.0
6615.9	2.7998	6628.7	7622.6	7626.2
6124.4	2.8983	6094.3	6979.8	6971.4
5683.6	2.9983	5616.8	6411.0	6392.9
5291.4	3.0985	5198.0	5908.8	5883.6
4946.7	3.1972	4827.1	5470.8	5440.2
4631.6	3.2978	4492.5	5075.0	5040.1
4350.9	3.3974	4195.7	4725.7	4688.3
4096.8	3.4971	3928.4	4411.8	4372.5
3865.3	3.5972	3686.7	4129.0	4088.7
3656.9	3.6961	3469.9	3876.4	3835.6
3465.5	3.7954	3271.4	3646.0	3605.0
3281.9	3.8993	3082.2	3426.8	3386.4
3120.3	3.9989	2916.5	3235.7	3195.7
2971.1	4.0984	2763.9	3060.6	3021.5
2827.7	4.2017	2618.0	2892.8	2855.2
2700.5	4.3004	2489.1	2745.6	2709.4
2580.8	4.4001	2368.1	2607.5	2572.8
2469.4	4.4994	2255.3	2479.6	2446.8
2363.8	4.6000	2150.2	2360.2	2329.3
2266.7	4.6988	2052.7	2250.1	2220.6
2173.8	4.7995	1960.1	2145.3	2117.9
2086.9	4.8999	1874.0	2048.9	2022.6
2006.4	4.9990	1794.5	1958.9	1934.7

[a] T is the temperature associated with the adjacent value of resistance for R_1, which was previously calibrated against T_{58} [2].

FIGURE 6. *Differences between the fitted functions,*

$$\log R = \sum_{n=0}^{3} A_n (\log T)^n$$

and the calibration data for resistors K, E, *and* I.

(a) See (a) of figure 4.
(b) The residual plots apply to the three resistors [K (.), E (+) and I (0)]; the calibration data for each resistor have been individually fitted to third order polynomials.

Similarities are to be detected in comparing the general characteristics of the coefficients in table 9. The third and fifth order polynomial fittings yield well determined coefficients while the fourth and sixth order coefficients, in some instances, leave much to be desired. The A_0's of the seventh order are generally not statistically significant.

The plots, for resistors K and E, in figures 6, 7, and 8 are representative of corresponding plots for resistors C, F, G, H, J, L, and M. That is, the systematic deviations (between the calibration data points [R_{data}, T_{data}] and the fitted functions [R_{calc}, T_{data}]) portrayed in figures 6 and 7 are most similar to residual plots of identical polynomial orders for the other seven resistors. The randomness of deviations for the seventh order polynomial (fig. 8) is also representative.

It is interesting to note that, with the exception of run VI, the corresponding coefficients for all of the resistors are generally of comparable magnitudes and identical signs. This observation is most pertinent to the coefficients of the third and fifth order polynomials, where the coefficients are properly significant; and, excepting A_0's, to the coefficients of the seventh order as well.
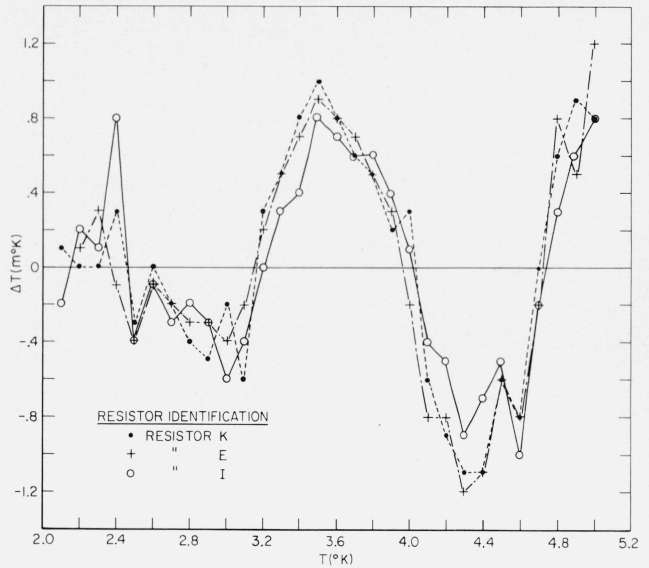

FIGURE 7. *Differences between the fitted functions,*

$$\log R = \sum_{n=0}^{5} A_n (\log T)^n$$

and the calibration data for resistors K, E, *and* I.

(a) See (a) of figure 4.
(b) The residual plots apply to the three resistors [K (.), E (+) and I (0)]; the calibration data for each resistor have been individually fitted to fifth order polynomials.
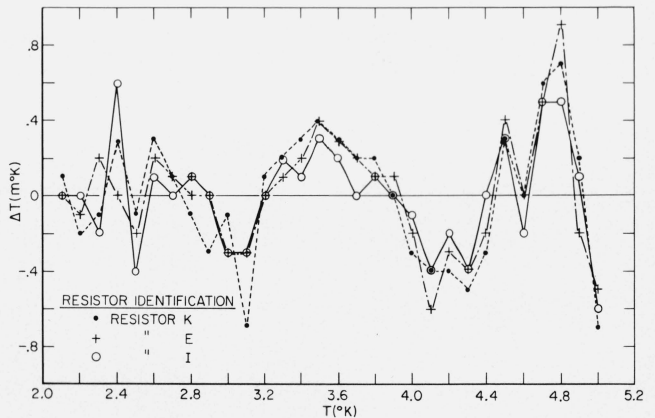
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TABLE 9. *Coefficients determined from fitting the polynomial* $\log_{10} R = \sum_{n=0}^{m} A_n (\log_{10} T)^n$ *to the calibration data of resistors*

| | Run III | | Run V | | | | Run VI | | | | Run VII | | | | Run IX | | | |
| | Resistor L | | Resistor H | | Resistor M | | Resistor C | | Resistor G | | Resistor F | | Resistor J | | Resistor K | | Resistor E | |
	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]	Coef.	δC [b]
$n=3$ A_0	5.367	(0.003)	5.328	(0.004)	5.280	(0.003)	5.330	(0.003)	5.317	(0.004)	5.388	(0.003)	5.398	(0.003)	5.319	(0.003)	5.439	(0.003)
A_1	−4.638	(.02)	−4.538	(.02)	−4.478	(.02)	−4.486	(.02)	−4.467	(.03)	−4.683	(.02)	−4.724	(.02)	−4.597	(.02)	−4.733	(.02)
A_2	3.522	(.04)	3.385	(.05)	3.492	(.03)	3.308	(.04)	3.196	(.06)	3.471	(.04)	3.466	(.04)	3.575	(.04)	3.542	(.04)
A_3	−1.685	(.03)	−1.573	(.03)	−1.737	(.02)	−1.537	(.03)	−1.467	(.04)	−1.611	(.02)	−1.556	(.03)	−1.752	(.03)	−1.667	(.03)
σ [a]	1.6×10^{-4}		1.8×10^{-4}		1.4×10^{-4}		8.5×10^{-5}		1.1×10^{-4}		1.4×10^{-4}		1.6×10^{-4}		1.6×10^{-4}		1.6×10^{-4}	
$n=4$ A_0	5.313	(.014)	5.285	(.017)	5.239	(.012)	5.385	(.014)	5.395	(.016)	5.343	(.012)	5.346	(.014)	5.271	(0.14)	5.385	(.014)
A_1	−4.186	(.11)	−4.174	(.14)	−4.136	(.10)	−4.969	(.12)	−5.164	(.14)	−4.308	(.10)	−4.293	(.11)	−4.200	(.11)	−4.282	(.11)
A_2	2.147	(.34)	2.276	(.43)	2.448	(.30)	4.884	(.40)	5.469	(.45)	2.329	(.29)	2.154	(.35)	2.364	(.35)	2.167	(.34)
A_3	[.135]	(.45)	[−.104]	(.56)	[−.355]	(.40)	−3.787	(.56)	−4.711	(.65)	[−.099]	(.39)	[.182]	(.47)	[−.147]	(.46)	[.154]	(.45)
A_4	−.885	(.22)	−.715	(.27)	−.672	(.20)	1.185	(.3)	1.709	(.34)	−.736	(.19)	−.845	(.23)	−.781	(.22)	−.886	(.22)
σ [a]	1.3×10^{-4}		1.6×10^{-4}		1.1×10^{-4}		6.2×10^{-5}		7.2×10^{-5}		1.1×10^{-4}		1.3×10^{-4}		1.3×10^{-4}		1.3×10^{-4}	
$n=5$ A_0	5.588	(.04)	5.578	(.06)	5.466	(.04)	5.422	(.084)	5.412	(.097)	5.531	(.05)	5.580	(.06)	5.506	(.05)	5.642	(.04)
A_1	−7.059	(.44)	−7.243	(.66)	−6.511	(.42)	−5.379	(.93)	−5.334	(1.1)	−6.273	(.48)	−6.733	(.58)	−6.662	(.55)	−6.968	(.46)
A_2	13.94	(1.8)	14.87	(2.7)	12.19	(1.7)	[6.683]	(4.0)	[6.26]	(4.7)	10.389	(2.0)	12.164	(2.3)	12.466	(2.2)	13.19	(1.9)
A_3	−23.65	(3.6)	−25.47	(5.4)	−19.98	(3.4)	[−7.677]	(8.7)	[−6.42]	(10.1)	−16.337	(4.0)	−19.984	(4.7)	−20.506	(4.5)	−22.05	(3.8)
A_4	22.69	(3.6)	24.41	(5.4)	18.77	(3.4)	[5.340]	(9.3)	[3.53]	(11.)	15.347	(3.9)	19.128	(4.7)	19.39	(4.5)	21.11	(3.7)
A_5	−9.198	(1.4)	−9.797	(2.1)	−7.580	(1.3)	[−1.753]	(3.9)	[−.78]	(4.6)	−6.270	(1.5)	−7.787	(1.8)	−7.864	(1.7)	−8.577	(1.4)
σ [a]	7.7×10^{-5}		1.2×10^{-4}		7.6×10^{-5}		6.4×10^{-5}		7.5×10^{-5}		8.6×10^{-5}		1.0×10^{-4}		9.9×10^{-5}		8.3×10^{-5}	
$n=6$ A_0	6.015	(0.2)	6.197	(0.31)	5.656	(0.21)	3.992	(0.37)	3.941	(0.48)	5.7	(0.24)	5.9	(2.8)	5.8	(0.27)	6.03	(0.22)
A_1	−12.43	(2.5)	−15.02	(3.9)	−8.89	(2.6)	13.69	(5.0)	14.27	(6.4)	−8.8	(3.)	−11.3	(3.5)	−10.3	(3.4)	−11.79	(2.7)
A_2	41.63	(13.)	54.96	(20.)	[24.49]	(13.5)	−98.1	(27.)	−105.5	(35.)	[23.]	(15.)	35.66	(18.)	[31.]	(17.)	38.06	(14.)
A_3	−98.72	(35.)	−134.2	(54.)	[−53.31]	(37.)	296.	(79.)	306.2	(101.)	[−52.]	(42.)	[−84.]	(49.)	[−71.]	(47.)	−89.49	(38.)
A_4	135.6	(52.)	187.8	(80.)	[68.87]	(55.)	−485.	(127.)	−501.1	(162.)	[69.]	(62.)	[115.]	(73.)	[95.]	(71.)	122.5	(57.)
A_5	−98.49	(41.)	−139.0	(63.)	[−47.20]	(43.)	416.	(108.)	429.3	(138.)	[−49.]	(49.)	[−83.]	(58.)	[−68.]	(56.)	−88.79	(45.)
A_6	29.05	(13.)	42.01	(21.)	[12.9]	(14.)	−147.	(38.)	−151.2	(49.)	[14.]	(16.)	[25.]	(19.)	[20.]	(18.)	[26.09]	(15.)
σ [a]	7.2×10^{-5}		1.1×10^{-4}		7.6×10^{-5}		4.7×10^{-5}		6.0×10^{-5}		8.6×10^{-5}		1.0×10^{-4}		9.8×10^{-5}		7.9×10^{-5}	
$n=7$ A_0	1.635	(0.5)	[1.780]	(1.4)	1.371	(0.66)	[3.8]	(2.3)	[4.3]	(3.0)	[1.396]	(0.9)	[.545]	(1.0)	[.305]	(0.8)	2.353	(0.9)
A_1	51.82	(7.5)	49.78	(20.)	53.96	(9.7)	[16.]	(36.)	[9.]	(46.)	54.79	(13.)	67.86	(14.6)	70.26	(12.)	42.10	(13.)
A_2	−357.2	(46.)	−347.2	(124.)	−365.7	(60.)	[−116.]	(238.)	[−68.]	(303.)	−371.3	(83.)	−455.6	(90.)	−469.	(75.)	−296.5	(77.)
A_3	1260.	(157.)	1236.	(419.)	1276.	(203.)	[362.]	(863.)	[183.]	(1100.)	1292.5	(280.)	1589.	(306.)	1633.	(256.)	1051.	(263.)
A_4	−2607.	(316.)	−2578.	(845.)	−2614.	(408.)	[−629.]	(1860.)	[−235.]	(2400.)	−2645.	(564.)	−3262.	(620.)	−3346.	(516.)	−2180.	(529.)
A_5	3184.	(378.)	3172.	(1010.)	3165.	(488.)	[601.]	(2390.)	[87.]	(3060.)	3200.	(674.)	3959.	(736.)	4053.	(616.)	2669.	(632.)
A_6	−2129.	(248.)	−2135.	(663.)	−2099.	(321.)	[−278.]	(1700.)	[91.]	(2200.)	−2122.	(443.)	−2633.	(484.)	−2691.	(405.)	−1788.	(415.)
A_7	601.3	(69.)	606.9	(185.)	589.	(89.)	[39.]	(510.)	[−73.]	(650.)	595.3	(123.)	740.7	(135.)	756.	(113.)	505.6	(116.)
σ [a]	3.4×10^{-5}		9.3×10^{-5}		4.5×10^{-5}		4.9×10^{-5}		6.2×10^{-5}		6.2×10^{-5}		6.8×10^{-5}		5.8×10^{-5}		5.9×10^{-5}	

[a] σ is the standard deviation computed from the residuals resulting from the polynomial least squares fitting.
[b] δC is the standard error of the adjacent coefficient.
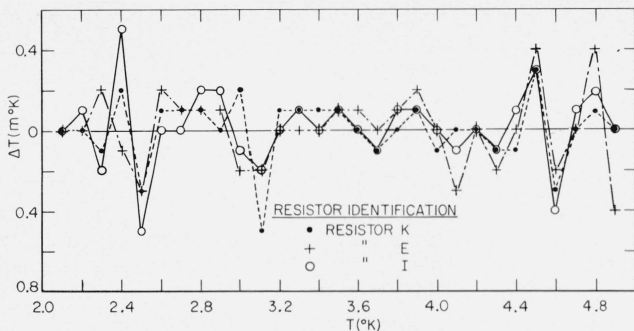[ ] Statistically, the bracketed coefficients are poorly determined.

FIGURE 8. *Differences between the fitted functions,*

$$\log R = \sum_{n=0}^{7} A_n (\log T)^n,$$

and the calibration data for resistors K, E, and I.

(a) See (a) of figure 4.
(b) The residual plots apply to the three resistors [K (.), E (+) and I (0)]; the calibration data for each resistor have been individually fitted to seventh order polynomials.

5.3. Analysis of the Control Resistor Data

The five individual calibrations of resistor I have been afforded a separate consideration since this resistor served as a "control" through all of the reported calibrations. Its values of resistance, when compared with the corresponding values for resistor #1 (or more appropriately temperatures derived from resistor #1), indicated that all calibrations of resistor I were consistant within ± 0.0007 °K. While no progressive drifting of the calibrations was observed (from Run III to IX), Run VII accounted for the largest portion of the maximum calibration deviation within the group of five runs; we are unable to ascertain specific causes for this occurrence.

The calibration data for resistor I have been functionally fitted in the same manner as was described in 5.1. Each of the five calibrations was treated independently and the resulting coefficients are listed in table 10. (The statistical uncertainty of each coefficient is within the adjacent parenthesis.) Since the coefficients of Run VI (table 9) are based upon calibration data that terminated at 4.2 °K, one must be cautious in comparing these coefficients with those of the other four calibration runs. Consequently, the general comparison between corresponding coefficients will be limited to Runs III, V, VII, and IX. Several conclusions are readily apparent.

(a) Third order fittings—the four coefficients are in excellent agreement (within, of course, the limits of

TABLE 10. *Coefficients determined from fitting the polynomial* $\log_{10} R = \sum_{n=0}^{m} A_n (\log_{10} T)^n$ *to the calibration data of resistor I*

	Run III Resistor	IδC [b]	Run V Resistor	IδC [b]	Run VI Resistor	IδC [b]	Run VII Resistor	IδC [b]	Run IX Resistor	IδC [b]	Combined data Resistor	IδC [b]
A_0	5.473	(0.0036)	5.473	(0.003)	5.459	(0.004)	5.474	(0.0027)	5.476	(0.0034)	5.473	(0.002)
A_1	−4.806	(.02)	−4.805	(.020)	−4.713	(.025)	−4.815	(.017)	−4.823	(.0206)	−4.806	(.011)
A_2	3.488	(.04)	3.487	(.039)	3.289	(.054)	3.508	(.033)	3.522	(.0409)	3.489	(.022)
A_3	−1.556	(.03)	−1.555	(.025)	−1.416	(.038)	−1.569	(.021)	−1.577	(.026)	−1.556	(.014)
σ [a]	1.7×10^{-4}		1.5×10^{-4}		1.0×10^{-4}		1.2×10^{-4}		1.6×10^{-4}		1.8×10^{-4}	
A_0	5.425	(.015)	5.426	(.013)	5.521	(.017)	5.441	(.012)	5.454	(.017)	5.438	(.008)
A_1	−4.404	(.13)	−4.42	(.11)	−5.265	(.15)	−4.540	(.10)	−4.639	(.14)	−4.512	(.070)
A_2	2.265	(.39)	2.31	(.34)	5.089	(.49)	2.671	(.31)	2.962	(.42)	2.590	(.21)
A_3	[.0632]	(.51)	[.0008]	(.44)	−3.985	(.7)	[−.460]	(.41)	[−.835]	(.55)	[−.363]	(.28)
A_4	−.787	(.25)	−.76	(.22)	1.354	(.37)	−.539	(.20)	[−.361]	(.27)	−.582	(.14)
σ [a]	1.4×10^{-4}		1.3×10^{-4}		8×10^{-5}		1.2×10^{-4}		1.6×10^{-4}		1.7×10^{-4}	
A_0	5.724	(.05)	5.710	(.04)	5.714	(.094)	5.653	(.046)	5.780	(.053)	5.714	(.035)
A_1	−7.539	(.51)	−7.386	(.38)	−7.407	(1.03)	−6.757	(.47)	−8.045	(.55)	−7.409	(.37)
A_2	15.14	(2.1)	14.48	(1.5)	14.48	(4.5)	11.765	(1.9)	16.936	(2.2)	14.50	(1.5)
A_3	−25.89	(4.2)	−24.52	(3.1)	−24.30	(9.8)	−18.782	(3.9)	−29.00	(4.5)	−24.43	(3.0)
A_4	24.94	(4.2)	23.53	(3.0)	23.05	(10.4)	17.607	(3.8)	27.54	(4.4)	23.31	(3.0)
A_5	−10.04	(1.6)	−9.47	(1.2)	−9.154	(4.4)	−7.074	(1.5)	−10.88	(1.7)	−9.34	(1.2)
σ [a]	9.0×10^{-5}		6.7×10^{-5}		7.2×10^{-5}		8.4×10^{-5}		9.8×10^{-5}		1.4×10^{-4}	
A_0	5.966	(0.25)	5.98	(.18)	4.069	(0.41)	5.60	0.24	6.09	.27	5.89	0.18
A_1	−10.57	(3.1)	−10.8	(2.3)	14.536	(5.4)	16.04	(3.0)	−11.98	(3.4)	−9.61	(2.3)
A_2	[30.78]	(16.)	32.0	(11.7)	−106.14	(30.)	[8.06]	(15.)	37.21	(17.4)	25.86	(11.7)
A_3	[−68.31]	(43.)	−71.9	(32.)	325.5	(86.)	[−8.73]	(41.)	[−84.00]	(47.0)	[−55.27]	(32.)
A_4	[88.7]	(65.)	94.7	(47.4)	−541.6	(138.)	[2.51]	(62.)	[110.2]	(70.5)	[69.8]	(47.)
A_5	[−60.5]	(51.)	[−65.8]	(37.4)	472.1	(118.)	[4.87]	(49.)	[−76.267]	(55.7)	[−46.]	(38.)
A_6	[16.4]	(17.)	[18.3]	(12.1)	−169.2	(41.)	[−3.88]	(16.)	[21.3]	(18.1)	[12.]	(12.)
σ [a]	9.0×10^{-5}		6.6×10^{-5}		5.1×10^{-5}		8.6×10^{-5}		9.8×10^{-5}		1.4×10^{-4}	
A_0	[0.193]	(0.5)	2.0915	(.51)	[2.7]	(2.5)	2.049	(1.1)	3.015	(1.3)	1.857	(0.90)
A_1	74.11	(7.4)	46.26	(7.4)	[35.]	(39.)	45.98	(15.)	33.18	(18.7)	49.60	(13.1)
A_2	−495.	(46.)	−322.	(46.)	[−244.]	(255.)	−314.8	(95.)	−243.2	(116.)	−342.1	(81.2)
A_3	1722.	(155.)	1134.	(155.)	[828.]	(926.)	1091.	(322.)	871.4	(394.)	1200.	(276.)
A_4	−3526.	(312.)	−2340.	(313.)	[−1628.]	(2000.)	−2217.	(649.)	−1819.	(793.)	−2467.	(557.)
A_5	4266.	(373.)	2849.	(374.)	[1870.]	(2568.)	2661.	(775.)	2234.	(948.)	2995.	(666.)
A_6	−2828.	(245.)	−1899.	(246.)	[−1160.]	(1819.)	−1750.	(509.)	−1499.	(623.)	−1990.	(438.)
A_7	793.	(68.)	534.	(68.)	[299.]	(548.)	4867.	(142.)	424.	(174.)	559.	(1.22)
σ [a]	3.4×10^{-5}		3.5×10^{-5}		5.2×10^{-5}		7.1×10^{-5}		8.9×10^{-5}		1.3×10^{-4}	

[a] σ is the standard deviation computed from the residuals resulting from the polynomial least squares fitting.
[b] δC is the standard error of the adjacent coefficient.
[ ] Statistically, the bracketed coefficients are poorly determined.

the stated statistical uncertainties). Furthermore, the ratio of an individual coefficient to its respective uncertainty indicates rather well defined coefficients, and the standard deviations of the individual least squares fittings are comparable.

(b) Fourth order fittings — while similarities exist between some coefficients, two features clearly indicate that this order of polynomial fitting presents no advantage. The A_3's are not defined, and the standard deviations, σ, show little improvement over the third order of fittings.

(c) Fifth order fittings — the coefficients, with respect to their statistical uncertainties, are clearly not as well defined as those of the third order polynomial; but the ratios are generally greater than six to one. More important, the standard deviations in polynomial fitting have materially improved over the third order thus indicating the superiority of the fifth degree for representing the calibration data. Here also, most of the corresponding coefficients i.e., A_0's, . . . A_5's, are nearly identical within the limits of their statistical uncertainties. Those from run VII exhibit noticeable divergencies; several paragraphs earlier we pointed out that the calibration data for run VII exhibited maximum variance with the other calibrations.

(d) Sixth order fittings — clearly, a sixth order polynomial produces no beneficial result.

(e) Seventh order fitting — while we should like to employ a seventh order polynomial for the data representation, the inconsistencies of the coefficients (A_0 and A_1 in particular) for run III clearly imply the possibility of its impropriety.

The polynomial coefficients for run VI, where the data is terminated at the upper limit of 4.2 °K, generally exhibit different qualitative trends, but are in quantitative agreement with the corresponding coefficients of the other runs for the fifth order polynomial. The general conclusion, which may be drawn from the comparison of the determined coefficients, strongly suggests that a fifth order polynomial offers a consistant representation for all of the calibration data (for resistor I) and that the standard deviation of such a fitting is about 9×10^{-5}. Deviations of the individual data points (Resistor I) from the fitted third order polynomial (run IX) are shown in figure 6. The curves' significant features are duplicated, within a spread of ± 0.0002 °K, for the corresponding representations of runs III, V, and VII. In the case of run VI, the general departures from the base line are reduced in size.

Figure 7, which portrays the deviations of individual data points (Resistor I) from the fitted fifth order polynomial (run IX), is also representative of the other complete calibrations of resistor I. While it exhibits systematic deviations, they are much less than those exhibited in figure 6. A comparable plot (figure 8), associated with the seventh order polynomial, exhibits a random distribution; although this condition is desirable, the indeterminacy of run III's seventh order coefficients causes us to hesitate in advocating use of the seventh order polynomial for representing this calibration data.
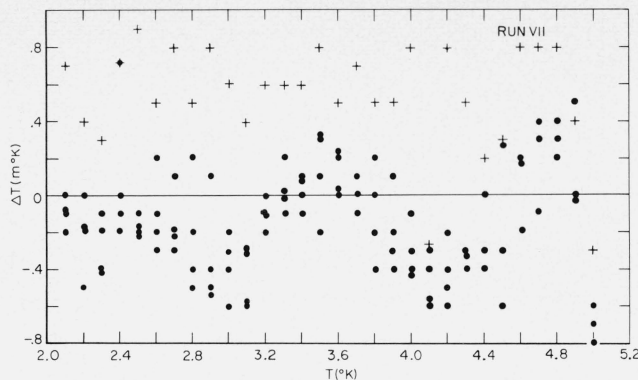

FIGURE 9. *Differences between the fitted function,*

$$log\ R = \sum_{n=0}^{5} A_n\ (log\ T)^n,$$

and all of the calibration data for resistor I.

(a) See (a) of figure 4.
(b) The residual plots apply to all of resistor I's calibration data (Runs III, V, VI, VII, and IX) which have been fitted to a fifth order polynomial. Residuals for Run VII are specifically identified (+).
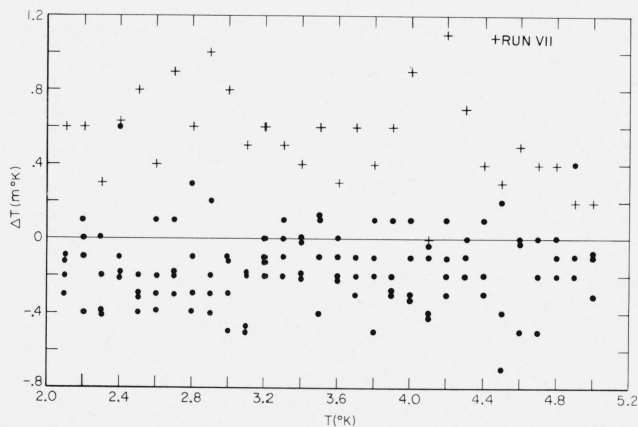

FIGURE 10. *Differences between the fitted function,*

$$log\ R = \sum_{n=0}^{7} A_n\ (log\ T)^n$$

and all of the calibration data for resistor I.

(a) See (a) of figure 4.
(b) The residual plots apply to all of resistor I's calibrated data (Runs III, VI, VII, and IX) which have been fitted to a seventh order polynomial. Residuals for Run VII are specifically identified.

In summary, the preceeding paragraphs have indicated that through the five calibration runs: the reference resistor #1 and the control resistor I have exhibited relative calibration constancies that are within a temperature equivalent of ± 0.0007 °K; additionally, for I, the third and fifth order polynomial coefficients are consistantly determined within the limits of their computed uncertainties.

Recognizing that the polynomial fittings of individual calibrations for I are somewhat restrictive in intercomparisons between different calibrations, we have "polyfitted" all of the resistor I data. The resulting coefficients, listed in the last column of table 10, are generally not significantly different (within the limits of the coefficients' uncertainties) from the coefficients of runs III, V, VII, and IX. Figures 9 and 10 are plots of deviations between the calibration data (R_{data}, T_{data}) and the fitted polynomials (R_{calc}, T_{data}) for the 5th and 7th order polynomials respectively. It is obvious in both plots that the data points from run VII consistently deviate from the rest of the calibration data and this accounts for the increased standard deviation that appears in the polynomial fittings of the last column. For each calibration temperature in figures 9 and 10 it is apparent that all of the calibrations agree within ± 0.0007 °K.

As a final criterion on the general validity of calibrations within the calibrator we have spot-checked the calibrations of run IX. Resistors I, K, and E were calibrated directly against helium-4 vapor pressures, in a constant temperature bath, at 2.2, 3.0, and 4.2 °K. Comparisons were made with the previous calibrations of run IX and the calibration differences (in temperature equivalents) are shown in table 11. Overall, the agreement between the calibrations is reasonable although the discrepancies at 4.2 and 3.0 °K for resistor I are somewhat large.

TABLE 11. *Intercomparison of calibrations from two methods* [a]

	Temperature difference		
4.2 °K	$mdeg$ [b] +1.4	$mdeg$ [b] +0.2	$mdeg$ [b] −0.75
3.0 °K	−1.2	0.0	−0.7
2.2 °K	+0.5	+0.2	−0.1
	Resistor I [c]	Resistor K [c]	Resistor E [c]

[a] Method A — the resistors were calibrated within the calibrator against resistor #1.
Method B — the resistors were calibrated within a "constant temperature" liquid helium bath against T_{58} [2] by means of vapor pressure measurements.
[b] The positive sign (+) means, for the same value of resistance, T (°K) of method B is greater than T (°K) of method A.
[c] Previous thermal histories indicated that, under 86 thermal cyclings between ambient temperatures and 4.2 °K, the 4.2 °K calibrations of resistors I, K, and E were reproducible to 0.8, 1.1 and 1.0 mdeg respectively.

6. Conclusions

We have shown that the resistance-temperature calibration (from 2.1 to 5.0 °K) for a particular group of ten similar germanium resistors can be represented by a polynomial function within the approximate limits of the calibration errors. In each case, based upon appropriate polynomial coefficients (fourth degree polynomials for resistors C and G, and fifth degree for the rest), generations of R, $\dfrac{dR}{dT}$ and $\dfrac{d^2R}{dT^2}$ as a function

of T (every 0.010 °K) have demonstrated that the determined polynomials are smooth (within the range 2.1 to 5.0 °K); the first and second derivatives do not change sign. While these conclusions are encouraging for the prospects of employing a fitted polynomial function to interpolate between calibration points, it should be stressed that the results apply to a select group of resistors. Also, caution should be employed in any attempts to use the polynomials for extrapolations beyond the calibration region.

The type of polynomial that has been used in this work,

$$\log_{10} R = \sum_{n=0}^{m} A_n (\log_{10} T)^n,$$

has been selected somewhat arbitrarily. There is no theoretical basis for selecting this polynomial to represent the resistors' calibration data, and the fitting of the polynomial has been an empirical procedure that is applicable to the limited range of calibration data. The values of the determined A_0's for any given resistor in table 10, indicate the inappropriateness of extrapolations based upon the determined polynomials. Additionally, when we have occasionally performed R, $\dfrac{dR}{dT}$ and $\dfrac{d^2R}{dT^2}$ generations (based on fitted polynomials) that extended beyond the calibration range, we found that the functions became erratic at external temperatures.

We believe that the polynomial, which we have employed, is not necessarily the most appropriate function for representing the calibration data of these resistors. However it does serve as an interpolation aid that can be cautiously used until a more meaningful representation is achieved. In the immediate future we shall seek to find improved functions not only for the data presented in this paper but for more extensive calibration data (2.3 to 20 °K) that we have previously published [1].

7. References

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